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Understanding Multiplication: Should We Memorise It or Not?

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Abstract. Multiplication might be considered as one of the basic mathematical facts that should be mastered as its major role is to help children understanding more advanced concept of mathematics. For example, multiplication could be found in fractions, irrationals, polynomials, vectors, and matrices. Some might suggest memorising multiplication facts may help children to develop their automaticity in computation. However, some others said understanding multiplication should be prioritised over memorisation since it will make the children capable to apply it when they work with complex problems. For this reason, examining whether children should memorise multiplication or not might be useful to help the learning process. In this article, I synthesise two theories which seem contradictory. The first theory is from Gray and Tall (1991) about proceptual understanding in mathematics and the second theory is dynamic instruction view from Byrnes and Wasik (1991). I found that these two theories can be made to complement each other in learning multiplication.

Keywords: Multiplication, Proceptual Understanding, Dynamic Instruction View.

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INTRODUCTION ~ Multiplication might be considered as one of the basic mathematical facts that should be mastered by children because multiplication has a major role in a more advanced concept of mathematics. Multiplication could be found in many areas such as in fractions, irrationals, polynomials, vectors and matrices (Davis and Renert, 2009). Therefore, it is crucial to master multiplication facts. Some might suggest that memorising it might help children to develop their automaticity in computation. Some others said that understanding multiplication should be prioritised over memorisation because it will make the children capable to apply it when they solve a complex problem. In this essay, I will firstly present my experiences which shape how I understand multiplication, then turning on to discussing an article which reshaped my perception about understanding mathematics. Lastly, I will continue to look into how multiplication should be understood in a more thorough way.

1. Personal Experience Dealing with Multiplication

When working with multiplication, I have utterly different experiences both as a learner and a teacher. In this section, I will describe those two different experiences which then influence my perception toward multiplication.

1.1. As a learner

When it comes to multiplication, I never had an idea that it is repeating addition until I was at a university level taking mathematics education major. I remember that when I was at primary school, I understood multiplication as something that should be memorised because my teacher asked me to memorise multiplication tables.

The multiplication tables were remembered at different grades. The higher the grade, the bigger the multiplication table would be. For example, when I was in the 3rd grade I had to memorise multiplication by 4-5 then, in the next grade I would remember multiplication by 6-8. Memorising was the only method that my teacher taught for multiplication. The way my teacher taught multiplication was just write the set of multiplications on the board then students would read it aloud together many times. It helped us to remember those sets of multiplication. After several days, my teacher would check our progress of this memorising by testing the students one by one in front of the class just for repeating multiplication sets we have memorised.

Later I found that the capability of memorising the multiplications helped me develop strong pre-knowledge and the easiness to understand a new concept. This happened when I had to deal with new teaching materials that needed multiplication as prior knowledge. Although I did not know that multiplication is a process of repeating addition however it seemed that it was not a big mistake because, in the next materials, multiplication was more likely to be used as a product of memorisation rather than a process.

1.2 As an educator

On the other hand, my previous role as a mathematics teacher brought me to a relatively distinctive experience to when I was a learner. Basically, I taught mathematics in a cramming school for students who needed additional time for learning outside their school time.

The quadratic equation was one of the worst materials my students could understand. Furthermore, when they were asked to figure out the quadratic equation solution, it was confusing for them because solving this problem required them to use both addition and multiplication concepts at the same time. Taking the quadratic equation $x^2 + 10x + 24 = 0$ as an example, the students were asked to solve it. This problem requires the students to define which two numbers have these consecutive results of +10 when they are added and +24 when they are multiplied. My students would find it a problem to decide which the two numbers are they since they might lack of memorising multiplication facts because they tend to do a repeating addition rather than having a memory of multiplication tables. Moreover, there are more than one pair of number which has a multiplication result of 24 $(12\times2, 3\times8$, and $6\times4)$. This might add to their confusion in solving the problem.

As I observed, many of my students might not have sufficient memory of multiplication sets. As reversed as me, some of them were more likely to understand multiplication as a process. I assume that the teacher who taught them multiplication might emphasise the learning of multiplication as a process. There was no duty for them to memorise the multiplication sets as I was in retrospect. Therefore, when they learned new material and it required them to do multiplication they would need a longer time to do it.

As I analysed my students from time to time, I found similarities. The problem my students faced in some cases was generated from their capability in memorising and understanding the ten multiplication sets. In a quadratic equation, for instance, they were unable to discover which digit multiplication

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they needed to solve the problem because they might understand multiplication as a process.

Taking my experience as both a learner and teacher into consideration, there is a clear difference in mathematics teaching practice especially in my country within these 10 years or so. I wonder about a balance teaching in delivering multiplication as both a process and something that should be memorised. I suspect either process or memorisation of multiplication should not be overlooked within the learning process. However, having a greater portion in imparting multiplication as something to be memorised might be more beneficial for students to help them understand the next materials.

2. Understanding Mathematics

For some mathematics education researchers, mathematical understanding is separated as procedural understanding and conceptual understanding (Gelman & Meck, 1986; Hiebert & Carpenter, 1992). Procedural understanding represents both procedure and computational competence while conceptual understanding is the capability to connect various concepts so that mathematical procedures become meaningful (Shimizu, 1996). However, there is still a long-standing and on-going dispute about the relations between procedural and conceptual understanding – which one supports the most for mathematical cognition (Rittle-Johnson, B., Schneider, M., & Star, J. R., 2015). If some mathematics education researchers might focus only on how children understand mathematics, an article by Gray and Tall (1991) entitled *Success and Failure in Mathematics: The Flexibility Meaning of Symbols as Process and Concept* has a different view that strikes me. They believe that their theory provides an extra ingredient in the discussion of mathematical understanding which is not merely defining it into a few types of understanding. This article helps me to reveal the mystery of why students who fail in mathematics will fail badly and fail more often.

2.1 Assertion

The article begins by looking at the concept of number to explain how it develops from process to concept. Gray and Tall interviewed children and found that there were many different methods children did in simple arithmetic. Then, they discovered that a mathematical symbol can be interpreted either process or concept. Finally, after reviewing further stages of the mathematical curriculum, they discovered this pattern appears in other mathematical concepts as well. For example,

" **3+2** is either the process of addition of 2 and 3 or the concept of sum,
3/4 can mean the process of division of 3 by 4 or the concept of fraction ¾,
+2 denotes the process of shifting 2 units to the right, and also the concept of signed number +2." (p.1-2)

Gray and Tall later reaffirmed that "a mathematical symbol could be seen either as an object that can be manipulated or a procedure to be carried out" (p. 2). Then, these two views are defined by them as a *procept* which refers to "a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either or both" (p. 2). Although the authors do not guarantee that all concepts in mathematics are procepts, however they argue that it appears in mathematics, particularly in arithmetic, algebra, calculus, and analysis.

2.2 Writers' point of view

Although it is not explicitly mentioned, for the reason below, I assume that Gray and Tall wrote this article from a constructivist view. Firstly, they referenced a notion from Richard Skemp that explained "*faux amis* - where the same word has very different meanings for different individuals" (p.1). They then refer to a mathematics symbol which also has an idea like "faux amis" which later they defined as procept. They claimed that the different perspectives in interpreting a mathematical symbol generate great flexibility or ambiguity which will differentiate students' capability in understanding mathematics. Therefore, this ambiguity might be their focus on their article. Because they want to concentrate on discussing different interpretations which then leads to the flexibility use of mathematics symbols, I suspect they have a view that children construct their knowledge from their perceptions.

Secondly, they claim that children who succeed in mathematics will learn how to develop new facts from the old in a flexible way. They then explained meaningful facts that could be decomposed and recomposed deliberately by a child would benefit them in deducing new facts effortlessly. From this argument, I assume that Gray and Tall believe that children also construct their knowledge based on their learning experience.

2.2 Writers' finding

After successfully revealing the subtle differences in mathematics, Gray and Tall later analysed and found that those who cannot notice this difference would be a failure in mathematics. They then refer to their previous work in 1991 which introduced the notion of *proceptual divide*, a gap between those who successfully recognise a mathematics symbol as a procept versus those who only interpret a symbol as a procedure.

The authors assured that this gap is occurred throughout the mathematics curriculum and the difference in thinking grows even wider between those who succeed and fail. They then hypothesised children who have *proceptual thinking* – using either procedures or concepts where appropriate/in a flexible way– will have what they say as "great mathematical power" (p. 4) and might become flexible thinkers.

Gray and Tall suspected that if there is great cognitive demand on a child, he/she might be successful previously and continuously ask "tell me how to do it" for ensuring the right procedure rather than trying to see the flexibility of procept. This would be the start on inevitable failure because Gray and Tall believed that failing at realising the procept will lead to failure in mathematics.

Taking multiplication as an illustration, Gray and Tall explained that if children fail to realise that multiplication is a procept, they will only see multiplication as a process of addition, then seeing the addition as a counting procedure. In this case, children will do double-counting process, doing the counting procedure for the addition process then repeating it to complete the process in multiplication.

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This will be more complex rather than if the children have proceptual thinking. That is why Gray and Tall concluded that "those who fail are doing a more difficult kind of mathematics compared to those who succeed" (p.1).

2.3 Exhortation

Sometimes children are given the freedom to construct their knowledge because it might give more meaningful understanding for them. In fact, this might generate a proceptual divide between learners because the symptoms of eventual failure could be disguised even though children could perform well in a computation or a procedure. Therefore, to lessen the proceptual divide, which hopefully would also prevent children from being inflexible thinkers, there is a need to diagnose whether students develop an appropriate strategy when they perform a process. Gray and Tall believed that the best way to do this is by discussing and listening to children explaining the process they carried out. I believe this process, in the context of school, as a confirmation between teacher and learner where teachers seek to know how their students' thinking was.

So, a key point which I take from this article is that teachers should not be happy if their learner can perform well in a mathematics problem/question. Rather they should find out the process used by their learners to achieve the "right answer". I would argue that it is important to make learners realise about procepts – without explicitly mention the word "procept" itself – so that they can attain ultimate success in mathematics.

2.4 Critic for the article

In my view, mathematical understanding that teachers want to be embodied within students' minds might not only end up in mastering proceptual thinking. At some points – in a multiplication context, memorisation might have a role after they could use procedures and concepts flexibly. For example, after students see multiplication as a procept, eventually they have to memorise multiplication tables to help them use it in a further stage of mathematics.

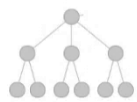
3. Possible Role of Memorisation in the Process, Concept, and Procept of Multiplication

In reflection on my experience in learning and teaching multiplication, I initially assumed that memorisation has a considerable role in mastering multiplication facts. However, after reviewing the work from Gray and Tall (1991), I then reconsidered the role of memorisation in multiplication when there is a fact that multiplication could be interpreted as either procedural knowledge or conceptual knowledge. Accordingly, in the search of the best possible practice in teaching multiplication, particularly in the context of Indonesian students, in this section I will discuss whether memorisation could be fitted in process, concept, or procept. As Gray and Tall (1991) assert that a concept is produced by a process, therefore I assume that, in the context of multiplication, procedural knowledge and conceptual knowledge come in order. Hence, I will divide the children's understanding of multiplication facts into three stages which are process stage, concept stage, and procept stage. Then, I will attempt to find the role of memorisation in each of these stages.

3.1 Memorisation in the Process stage

This step might be relevant to procedural knowledge in which children might see multiplication as a process of *repeating addition* (Gray and Tall, 1991). However, Davis and Renert (2009) discovered that the multiplication process could also be seen as *grouping, branching or folding* (see Table 1). Later in their explanation, Gray and Tall (1991) also said that the symbol 3x4 can be *3 lots of 4* or *3 multiplied by 4*. The former is 4+4+4 and the latter is 3+3+3+3. Although these two processes have the same results however the processes themselves are different. Then, I conclude that there might be many different processes in multiplication and I would divide these identified processes into two categories which are multiplication as *n lots of m* and multiplication within a context described below.

Table 1. Different multiplication processes

Process	Meaning
Grouping	4x5 means either 4 sets of five or 5 sets of four
Branching	2x3 means 
Folding	2x3 means do a 2-fold, then a 3-fold, giving 6 regions

3.1.1 Multiplication as *n lots of m*

If we are discussing about students in public schools in Indonesia, multiplication is introduced as repeating addition as some examples below

1x1=1	1x2=2
2x1=1+1	2x2=2+2
3x1=1+1+1	3x2=2+2+2
4x1=1+1+1+1	4x2=2+2+2+2
5x1=1+1+1+1+1	5x2=2+2+2+2+2
.	.
.	.
10x1=1+1+1+1+1+1+1+1+1+1	10x2=2+2+2+2+2+2+2+2+2+2

The processes above are what we know as *n lots of m*. I assume that our students in public schools will know about this idea as it is the method commanded in our curriculum for introducing multiplication. Therefore, I believe that the students would have the same interpretation of multiplication as n lots of m. This process would not change, in other words it is the syntax as repeating addition. For example, at primary level, 2x5 will always be seen as 2 lots of 5 and 5x7 will always be seen as 5 lots of 7. Because there is a pattern, then this could be a possibility that children will memorise the syntax of repeating addition. For the purpose of coherence, I will use the term static

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process to represent multiplication as *n-lots-of-m* process.

3.1.2 Multiplication within a context

Some possible meanings of multiplication processes such as grouping, branching, and folding (see Table. 1) would be categorised as process within a certain context. In contrast to the static process, I assume that the process of multiplication may be different depending on the topic discussed. For example, in arithmetic, multiplication can be seen as grouping while in two-dimension topics, multiplication can be seen as a process of folding. However, some teachers might not be so explicit about the idea of multiplication embedded in a topic hence their students might be unaware or not be able to identify other possible processes of multiplication. Moreover, each of these processes would have a different process in a different interpretation, therefore this kind of processes can be work out without memorisation.

To conclude, considering that there might be a syntax of repeating addition to be memorised in the static process, I would expect that memorisation might have a role in the process stage.

3.2 Memorisation in the Concept stage

At this stage children will find a new fact of multiplication as a product of repeating addition they have known. Then, the new facts are what we know as the multiplication facts –will be used interchangeably with the multiplication concept – or the multiplication table from 1 to 10. As one of the basic mathematical facts, multiplication facts are considered to be important to understand the next level of mathematics. Therefore, fluency in using multiplication facts would support children in the next-level performance. For example, single-digit multiplication proficiency will support the understanding of a multi-digit multiplication. I could see this phenomenon in my own experience as a learner. I barely find any difficulty in computation because I have memorisation of single-digit multiplication. This indicates that there is a possibility to memorise the multiplication table in order to achieve mathematical computational fluency which is a crucial component of mathematical power development (Fuson, 2003).

3.3 Memorisation in the Procept stage

In the previous stage, I suspect that memorisation might be required in the static process of multiplication. Then, considering that a procept in multiplication consists of both a process of repeating addition and a multiplication concept, hence it is easy to suspect that there should be something to be memorised in procept. However, in this stage I would refer to the attainment of proceptual thinking. Proceptual thinking in multiplication means that children can use multiplication as a repeating addition where appropriate and use multiplication concepts where appropriate. Therefore, what I want to refer as procept in this stage is the capability to recognise the ambiguity of multiplication as a process or a product. This kind of awareness, I believe, is not feasible to be memorised. Consequently, there is no rule of memorisation in the procept stage.

4 Is Memorisation Truly Required for Multiplication?

In the former discussion, we have found that there are two possibilities in which memorisation takes part, in static process and concept stage. In this section, I will examine these assumptions based on my experience in teaching and learning multiplication and some other theories.

Unlike the process of branching and folding, in static process, I assume that the procedure of multiplication might not be correlated with any topics as it could be an initial step to introduce multiplication. According to Gray and Tall (1991), this process would lead to the development of the multiplication concept. Therefore, I assume that the static process does not require memorisation in the case of nurturing flexible thinking. Based on my hypothesis, after attaining the concept of multiplication itself, children might find other processes of multiplication which are more connected to different topics as I explained previously in the multiplication within a context. In short, conceptual knowledge will help children to see different procedures of multiplication (procedural knowledge derives from conceptual knowledge). However, this idea seemed contradicting to Gray and Tall because they assert that a process of repeating addition will develop a concept of multiplication (conceptual knowledge derives from procedural knowledge). I will describe this inconsistency later in the last section.

Turning our attention to the concept stage, multiplication is a part of the mathematics curriculum that can be worked out by children themselves from what they already know (repeated addition) and they can examine it for correctness (Hewitt, 1999). Therefore, there is no demand for memorising multiplication because memorising it could be unnecessarily damaging, without the ability to use numbers flexibly (Boaler, 2015). Boaler said that instead of memorising, multiplication could be performed by children providing that they have excellent number sense. For example, children might have memory of a multiplication set of either 7 or 8, when children are asked to solve 7 x 8, a child with number sense would capable to solve 7 x 7 is 49 then adding it by 7 to make 56. Another method is that subtracting two 7's from ten 7's (70 - 14). When it comes to number sense, Boaler seems to be a proponent of Gray and Tall (1991) because Boaler believes that students with number sense could see number in a flexible way. Moreover, discussing multiplication as a conceptual category, Haapsalo and Kadijevich (2000) suggested that conceptual knowledge generally requires conscious thinking which means it requires consciousness of the applied actions and knowledge of why they work. I assume that this is related to the context of concepts which could have different meanings even though they share the same symbols. However, Wallace & Gurganus (2005) propose that although memorising multiplication might be important for children, it should be based on an understanding of the operations as well as thinking strategies. Therefore, children do not necessarily require memorisation as multiplication could be worked out from what they have known, apart from remembering order and names for numbers as arbitrary facts in mathematics (Hewitt, 1999).

In the reflection on my experience, as a learner, sometimes I could see that memorising multiplication tables is terrifying because I have to remember many things. However, after that I could perform well in computation. On the other hand, I acknowledge that my number sense in multiplication was not

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particularly good because I habitually relied on my memorisation. Moreover, if children do not have number sense, they might be stuck in the repeating addition process such as I found in my students' learning. Nevertheless, since a demand for memorising may trigger mathemathics anxiety to some children (Morris, 1981), it is helpful to minimise the use memorisation. Therefore, I would suggest that for reducing mathematics fear towards multiplication, children could start learning some multiplication tables which are relatively easy to be worked out such as 1,5, and 9. These could be utilised as a bridge to work on other single-digit multiplications using number sense. Then, so as to reach automaticity, it will be more beneficial for the children to do regular practices (Carnine, 1997).

5 Understanding Multiplication is a Long Process: Confirming the Contradiction

Referring to the section 2, I suspect that there is an inconsistency, linked to whether procedural knowledge or conceptual knowledge that comes first. Gray and Tall (1991) assert that a procedural knowledge would produce a conceptual knowledge. In contrast, there is a Dynamic Interaction View (Byrnes and Wasik, 1991) which suggests that "conceptual knowledge forms a basis on which new procedures are acquired" (p. 785). The latter view suggest that at first, various examples of problem solving which accommodate radical constructions from children and have a rich conceptual knowledge are presented to develop procedures for a specific task. This can be reached by making discriminations and generalisations as a capability of when and where to apply a procedure. For instance, as 2x3=3+3 or 2+2+2, children can decide using the former or the latter process appropriately when it is needed. Later, this will develop procedures for a specific task that will become automatic for children. This step is then known as *proceduralisation*.

At first these two views seems contradict, however if we analyse meticulously, the notion of procept is similar to proceduralisation. Therefore, I hypothesise that the dynamic interaction view might supplement the Gray and Tall theory. Accordingly, in Figure 1, I attempt to illustrate how children develop their understanding in multiplication by adopting Gray and Tall's theory and the dynamic interaction view.

From this relation, I would conclude that understanding multiplication is a long process. As students at primary stage might not have many contexts to the approach of multiplication, I would assert that, initially, children might be taught multiplication in the approach of Gray and Tall. However, it might be important to convey to the children that "it is not the only way you can see multiplication" therefore the children do not have a fixed mind of multiplication. In the later stage, on the other hand, we can adopt the dynamic interaction view to encourage the awareness of the children toward the idea of multiplication when they work with other contexts or an advanced concept. Hence, the children could make a connection in which I believe proceduralisation will take place. By doing so, children could evolve their proceptual thinking in multiplication.

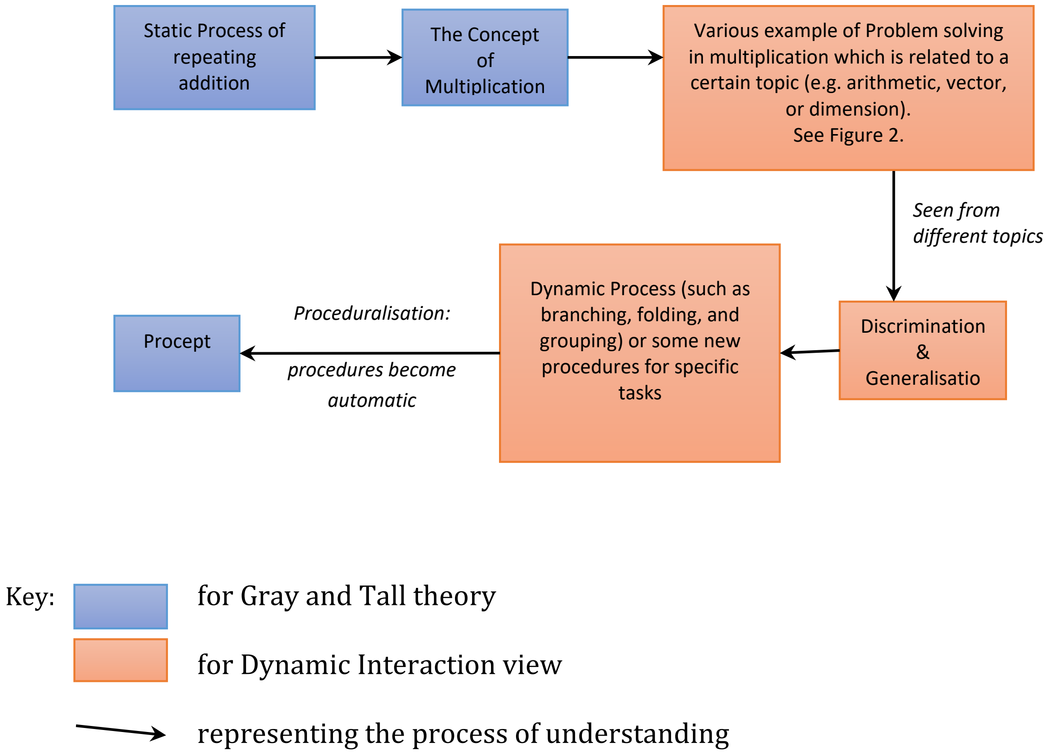

Figure 1. Multiplication Approach

Multiplication as a grouping process
My mom is making 3 pots of potato soup. She wants to put 4 potatoes in each pot. How many potatoes does she need?
Multiplication as a repeating addition
Erna has two mangoes, Ginta has two apples, and Andi has two oranges. How many fruits do they have together?

Figure 2. Examples of Multiplication Problem in Arithmetic topic

CONCLUSION

I assume that learning multiplication should not be memorised instead of teaching children to have number sense enabling them working out with multiplication. In my view, imparting understanding of multiplication is complex. This does not come to an end even when the children already hold the idea of multiplication. However, in many teaching and learning practices, it seems impossible to develop the idea of multiplication, having assumed that mathematics is generally imparted in segments. Whereas, I presume, by understanding the connection, students could be able to not only do the right thing but also know the reason for it. This is what I believe makes mathematics meaningful. Whether or not my assumption works for children

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understanding, the multiplication approach presented in this essay might worth to be hold by a teacher so that he can present a connection in teaching multiplication.

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